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# WEIERSTRASS FACTORIZATIONS IN COMPACT RIEMANN SURFACES

BY

PASCUAL CUTILLAS RIPOLL* AND JOSÉ MARÍA VERDE RAMÍREZ

*Department of Pure and Applied Mathematics, University of Salamanca, Spain*
*e-mail: pcr@gugu.usal.es*

ABSTRACT

Let $\mathcal{V}'$ be the complementary in a compact Riemann surface $\mathcal{V}$ of a point (or a finite set). In this paper are characterized the subfields, of the field of meromorphic functions in $\mathcal{V}'$, containing sufficient functions to verify a factorization property, similar to that of the classical Weierstrass theorem. It is also seen that the field generated by the Baker functions is not of this type, and the problem is solved of determining the divisors, in $\mathcal{V}'$, of the holomorphic functions admiting Weierstrass factorizations with Baker functions as factors. As an application, a theorem is obtained characterizing the infinite products, of meromorphic functions in $\mathcal{V}$ with bounded degree, which converge normally in $\mathcal{V}'$.

## 1. Introduction, notations and preliminary results

Let $\mathcal{V}$ be a compact Riemann surface of genus $g$, $\infty$ be a point of $\mathcal{V}$, and $\mathcal{V}' = \mathcal{V} - \{\infty\}$. Let $(p_n)$ be a sequence in $\mathcal{V}'$ converging in $\mathcal{V}$ to $\infty$. Then it is well known that, for every $n \in \mathbb{N}$, there exists a holomorphic function $f_n$ in $\mathcal{V}'$, with a simple zero at $p_n$ , without zeros in $\mathcal{V}' - \{p_n\}$, and such that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$. We shall call such a product, a Weierstrass product associated with the sequence $(p_n)$, and the above statement will be expressed in short by saying that the field $M(\mathcal{V}')$, of meromorphic functions in $\mathcal{V}'$, verifies the WF (abbreviation of Weierstrass factorization) property, i.e. for every sequence $(p_n)$ as above there is a Weierstrass product with factors in $M(\mathcal{V}')$ associated with

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$(p_n)$. Furthermore, if $V$ is an open neighbourhood of $\infty$ in $\mathcal{V}$ with a coordinate $z$ such that $z(\infty) = 0$, and if the points $p_n$ belong to $V$, it is also known (see, for instance, Baker [1]) that for every $n \in \mathbb{N}$, $f_n$ can be taken to be in $V - \{\infty\}$ of the form $he^{P(1/z)}$, with $h$ meromorphic in $V$ and $P$ a polynomial. Therefore, if $\overline{K}_\infty$ is the field generated by all functions in $\mathcal{V}'$ of this form, $\overline{K}_\infty$ verifies the WF property in $\mathcal{V}'$ too.

Consider now the subfields of $\overline{K}_\infty$ , or in general of $M(\mathcal{V}')$, which also contain sufficient functions to verify the WF property, and note that if $K$ is such a field then for every $p \in \mathcal{V}'$, at least a function with divisor $p$ must belong to $K$, and that (if $\infty$ is not a Weierstrass point) the simplest subfield of $\overline{K}_\infty$ verifying this, is the field $K_\infty$ generated by the so-called Baker functions in $\mathcal{V}'$, i.e. the functions of the above said type corresponding to polynomials $P$ of degree $\leq g$ (see Cutillas [2]). These considerations together with the previous ones lead to the following natural questions, in whose statement the expression "WF-field" means a subfield of $M(\mathcal{V}')$ containing the field $M(\mathcal{V})$ of meromorphic functions in $\mathcal{V}$, and verifying the WF property:

(1) Is $K_\infty$ also a WF-field in $\mathcal{V}'$?

(2) If the answer to (1) is negative, can we determine the Weierstrass products with factors in $K_\infty$ (i.e. with Baker functions as factors)?

(3) Can we characterize all the WF-fields in $\mathcal{V}'$?

The purpose of the authors in this paper is to study WF-fields and, in particular, to provide solutions to these problems. For instance, we shall see that the answer to question (1) is negative by showing a certain condition on the sequence $(p_n)$, which is equivalent to the existence of an associated Weierstrass product with factors in $K_\infty$, and we shall characterize the WF-fields in $\mathcal{V}'$ by means of a simple criterium, which also allows us to determine those among them which are generated by functions with finite divisor (observe that every WF-field in $\mathcal{V}'$ contains one of this type). Specifically, in Section 2 we prove that the existence of an associated Weierstrass product $\prod_{n=1}^{\infty} f_n$ , with factors in $K_\infty$ , is equivalent to the absolute convergence of the series $\sum_{n=1}^{\infty} z(p_n)^{g+1}$, and we draw some interesting consequences from the hypothesis of the convergence of the analogous series with exponent 1. Part of the obtained results are generalized later: on one hand by allowing larger orders for the poles at $\infty$ of the $d \log f_n$, and, on the other, by considering pairs of sequences of positive finite divisors with bounded degree in $\mathcal{V}'$, from which we deduce as an application a result characterizing the infinite products, of functions in $M(\mathcal{V})$ with bounded degree, which converge normally in $\mathcal{V}'$. Finally, in Section 3, some equivalent properties to that of being a WF-field are shown: one of them justifying the terminology used, and the other leading to the determination of all WF-fields in $\mathcal{V}'$ generated by functions with finite divisor (by means of the exponentials of holomorphic functions belonging to them and certain natural group homomorphisms), as well as to the proof that there is no minimal WF-field, after which we finish with a note remarking that these results can be easily generalized to complementaries of arbitrary nonempty finite subsets of $\mathcal{V}'$.

$\mathcal{V}$, $\infty$, $\mathcal{V}'$, $K_\infty$ , $\overline{K}_\infty$ , $M(\mathcal{V})$, $M(\mathcal{V}')$, $V$ and $z$ will be as above. $V'$ will be $V-\{\infty\}$, and we shall suppose further that $V$ is a coordinate disk with respect to $z$ (i.e. $\overline{V}$ is contained in a coordinate open neighbourhood of $\infty$, with coordinate $z$, and $z(\overline{V})$ is a closed disk in $\mathbb{C}$). **A coordinate disk in** $U$, for any open subset $U$ of $V$ which contains $\infty$, will mean a coordinate disk, $D$, with respect to $z$, centered at $\infty$ (in the obvious sense), and such that $\overline{D}$ is contained in $U$.

$G_\infty$ , $\overline{G}_\infty$ and $G(\mathcal{V}')$ will be respectively the groups of functions in $K_\infty$, $\overline{K}_\infty$ and $M(\mathcal{V}')$, having finite divisor. Unless otherwise stated, we shall suppose that $\infty$ is not a Weierstrass point and that the genus $g$ of $\mathcal{V}$ is $> 0$. For every open subset $U$ of $\mathcal{V}$ or any other Riemann surface, $O(U)$ will be the ring of holomorphic functions in $U$, $M(U)$ the ring of meromorphic functions in $U$, and $M^*(U)$ the multiplicative group of $M(U)$.

$A_1, \ldots, A_g, B_1, \ldots B_g$ will be piecewise $C^1$ curves in $\mathcal{V}'$ defining a canonical system of generators for the fundamental group of $\mathcal{V}$ (following, for instance, the terminology in Gunning [4]), and $\Delta$ will be the simply connected open subset of $\mathcal{V}$ complementary of the union of these curves. We shall suppose without loss of generality that the topological closure $\overline{V}$ of $V$ is contained in $\Delta$, and we shall put $\Delta' = \Delta - \{\infty\}$.

For every nonvoid finite subset $S$ of $\mathcal{V}$, $A(\mathcal{V}-S)$ will be $O(\mathcal{V}-S) \cap M(\mathcal{V})$, and when we consider $A(\mathcal{V}-S)$ (or any other space of holomorphic functions in an open subset of a Riemann surface) as a topological space, the topology will be that of the uniform convergence in compact subsets.

The following theorem, which will be useful later, is an easy consequence of theorem 10 in Royden [7].

THEOREM 1.1: *Let $U$ be a nonvoid open subset of $\mathcal{V}$, different from $\mathcal{V}$, and $S$ be a finite subset of $\mathcal{V}-U$ containing exactly one point of each connected component.*

*Then $A(\mathcal{V}-S)$ is dense in $O(U)$.*

As we have said in the Introduction, there is a long-time generalization of the Weierstrass factorization theorem, for holomorphic functions in $\mathcal{V}'$ (which seems to have been proved for the first time by Günther [5]), in which the elementary factors of the complex plane version are replaced by functions having singularities of polynomic exponential type at $\infty$ (i.e. functions of $\overline{G}_\infty$). Besides the paper of Günther, and the said book of Baker, a more modern proof can be either seen later in Section 3, since it is a particular case of an implication in a theorem characterizing WF-fields which will appear there, or obtained directly from Theorem 1.1 by reasoning as in that theorem. This result will be utilized in Section 2, and can be stated in the following way:

THEOREM 1.2: *$\overline{K}_\infty$ is a WF-field in $\mathcal{V}'$.*

## 2. Weierstrass products with factors in $K_\infty$

First, we explain some more terminology to be used from now on.

*Definition 2.1:* Given a sequence $(p_n)$ in $\mathcal{V}'$ converging in $\mathcal{V}$ to $\infty$, consider the positive infinite divisor in $\mathcal{V}'$, $\delta = \sum_{n=1}^{\infty} p_n$. A Weierstrass product associated with $\delta$ will be an infinite product normally convergent in $\mathcal{V}'$ of the type $\prod_{n=1}^{\infty} f_n$, where $f_n$ is, for every $n \in \mathbb{N}$, a function in $O(\mathcal{V}')$ having $p_n$ as divisor.

A Weierstrass product with factors in a subfield $K$ of $M(\mathcal{V}')$ will be a product $\prod_{n=1}^{\infty} f_n$ as above such that all the functions $f_n$ belong to $K$.

Note that, with the notation of this definition, a Weierstrass product associated with $\delta$ is the same as a Weierstrass product associated with the sequence $(p_n)$ in the sense of Section 1.

We also recall, since it seems to be not totally standard, the following definition:

Given an infinite product $\prod_{n=1}^{\infty} F_n$, of meromorphic functions $F_n$ in some Riemann surface $\mathcal{W}$, it is said to be normally convergent in $\mathcal{W}$, if for every compact subset $K$ of $\mathcal{W}$ there exists $n_0 \in \mathbb{N}$ such that:

(1) If $n \geq n_0$, then $F_n$ has no pole in $K$ and $\|F_n - 1\|_K < 1$, with the habitual notation denoting the supreme norm.

(2) $\sum_{n=n_0}^{\infty} \|\log(F_n)\|_K < +\infty$, where $\log(F_n)$ represents the principal branch of the logarithm of $F_n$ (i.e. the branch whose imaginary part takes values in $(-\pi, \pi)$).

In order to study Weierstrass products, it will be convenient, as we shall see later, to consider convergent series of differentials.

Given a series $\sum_{j=1}^{\infty}\theta_j$, where the $\theta_j$ are meromorphic differentials in $\mathcal{W}$, with $\mathcal{W}$ as above, we shall say that it converges normally in $\mathcal{W}$ (resp. uniformly in compact subsets of $\mathcal{W}$) if for every compact subset $K$ of any coordinate open subset $W$ of $\mathcal{W}$, with coordinate $w$, there exists $j_0 \in \mathbb{N}$ such that $\theta_j$ has no pole in $K$ for every $j \geq j_0$, and such that $\sum_{j=j_0}^{\infty}\|\theta_j/dw\|_K$ converges (resp. $\sum_{j=j_0}^{\infty}\theta_j/dw$ converges uniformly in $K$), where of course $\theta_j/dw$ represents the meromorphic function $h_j$ such that $\theta_j = h_j dw$ in $W$.

We shall also use the analogous terminology for infinite series of functions.

The following conventions will be useful for the sake of simplicity. It will be always evident that they mean no loss of generality.

WARNING 2.2:

(1) *In the sequel, whenever we consider a series having possibly a finite number of undefined terms (for instance $\sum_{n=1}^{\infty}z(p_n)$, with $(p_n)$ as in Definition 2.1) and we say that this series converges in some sense, it must be understood that by getting rid of that finite set of terms, the resulting series converges in the indicated way. The same will be valid for infinite products.*

(2) *Every infinite divisor in $\mathcal{V}'$ (resp. every sequence of finite divisors in $\mathcal{V}$ tending to $\infty$, in the obvious sense) will be supposed, if necessary, to be supported (resp. to have all its terms supported) in $V$.*

Let us consider again a sequence $(p_n)$ as in Definition 2.1, and the divisor in $\mathcal{V}'$, $\delta = \sum_{n=1}^{\infty}p_n$. We want to study in this section the problem of finding out, among all possible sequences $(p_n)$, those for which there exists a Weierstrass product, with factors in $K_\infty$, associated with the corresponding $\delta$. We begin with an observation, which is trivial, but will be fundamental.

*Remark 2.3:* Let $\delta = \sum_{n=1}^{\infty}p_n$ be as above, and $\phi$ be a function in $O(V)$ having a zero of order $\geq \ell \in \mathbb{N}$, at $\infty$. If $\sum_{n=1}^{\infty}|z(p_n)|^\ell$ converges, then the same is true for $\sum_{n=1}^{\infty}|\phi(p_n)|$.

As a consequence, the condition on the series $\sum_{n=1}^{\infty}z(p_n)$, of being absolutely convergent, is independent of $V$ and $z$ (if $V$ and $z$ are as in the Introduction).

The next proposition is useful for simplifying proofs of existence of Weierstrass products. In its statement and often in the sequel, we shall utilize, for a certain type of function, the denomination introduced, for brevity reasons, in the following:

*Definition 2.4:* Given a function $f \in M(\mathcal{V}')$, we shall say that it is $\Delta$-simple, if its divisor is supported in $\Delta$ (i.e., all its zeros and poles are in $\Delta$), and if the integrals of $d\log f$ along $A_1, \ldots, A_g, B_1, \ldots, B_g$ are null.

PROPOSITION 2.5: *If $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$, with $f_n \in G(\mathcal{V}')$ for every $n \in \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $f_n$ is a $\Delta$-simple function.*

*Proof:* Let $f \in M(\mathcal{V}')$ be the uniform limit on compact subsets of $\mathcal{V}'$ of the sequence $(\prod_{j=1}^{n} f_j)$. We can suppose without loss of generality that the divisor of $f$ is supported in $\Delta'$, and that the same is true for all functions $f_n$. Then, since $\int_{A_j} d\log f$ and $\int_{A_j} d\log f_n$ are, for every $n \in \mathbb{N}$, integral multiples of $2\pi$i, and since $\int_{A_j} d\log f$ coincide with the sum of the series $\sum_{n=1}^{\infty} \int_{A_j} d\log f_n$, for $j = 1, \ldots, g$, one deduces easily that all except possibly a finite number of summands in this series are zero. Being, finally, the same reasoning evidently valid for the curves $B_j$, the proof is finished. ∎

Before stating a first auxiliary result, it is convenient to explain some more notation:

$\{\omega_1, \ldots, \omega_g\}$ will be a basis of the space of holomorphic differentials in $\mathcal{V}$ verifying $\int_{A_i} \omega_j = \delta_{ij}$ (Kronecker's delta).

For $j = 1, \ldots, g$, $\theta_j$ will be the unique holomorphic differential in $\mathcal{V}'$ such that $\theta_j - dz/z^{j+1}$ is holomorphic in $V$, and such that $\int_{A_\ell} \theta_j = 0$, for every $\ell = 1, \ldots, g$.

For every pair $a, b$ of points in $\Delta$, $\theta_{ab}$ will be the normal differential of the third kind associated with $a, b$, and with the system of curves $A_1, \ldots, A_g, B_1, \ldots, B_g$; i.e. $\theta_{ab}$ is holomorphic in $\mathcal{V} - \{a, b\}$, has simple poles at $a, b$ with residues $1, -1$, respectively, and $\int_{A_\ell} \theta_{ab} = 0$, for $\ell = 1, \ldots, g$.

Consider now the functions $\phi_\ell$ of $p \in \Delta'$, defined for $\ell = 1, \ldots, g$, by

$$\phi_\ell(p) = \frac{1}{2\pi i} \int_{B_\ell} \theta_{p\infty} = \int_{\infty}^{p} \omega_\ell$$

(see, for instance, Farkas–Kra [3]), where the integral with limits $\infty$ and $p$ is taken along a curve in $\Delta$. The last equality implies that each one of these functions can be also considered as a holomorphic function in $\Delta$ taking the value 0 at $\infty$. Let $\varphi_j \colon \Delta' \to \mathbb{C}$ be, for $j = 1, \ldots, g$, such that $\theta_{p\infty} + \sum_{j=1}^{g} \varphi_j(p)\theta_j$ has null integrals along $B_1, \ldots, B_g$, for every $p \in \Delta'$. Then, integrating along $B_\ell$, one has:

$$2\pi i\phi_\ell(p) + \sum_{j=1}^{g} \varphi_j(p) \int_{B_\ell} \theta_j = 0, \tag{2.5.1}$$

for $\ell = 1, \ldots, g$. Hence, since the matrix with coefficients $\int_{B_\ell} \theta_j$ is invertible because of our assumption on the point $\infty$, we deduce that $\varphi_1, \ldots, \varphi_g$ are likewise holomorphic functions in $\Delta$ with a zero at $\infty$ (they form, as well as the functions $\phi_1, \ldots, \phi_g$, a set of $g$ linearly independent integrals of the first kind in $\Delta$, in the classical sense).

Note also, for later use, that if $p$ is a point of $\Delta'$ and if $f_p$ is a $\Delta$-simple function in $G_\infty$ with divisor $p$, then with the notation of above,

$$d \log f_p = \theta_{p\infty} + \sum_{j=1}^{g} \varphi_j(p)\theta_j. \tag{2.5.2}$$

LEMMA 2.6: *For every $j \in \{1, \ldots, g\}$, the sum $\varphi_j + z^j$ has a zero of order $\geq g+1$ at $\infty$.*

*Proof:* Since

$$\int_{B_\ell} \theta_j = \frac{2\pi i}{j} \lambda_{\ell,j-1},$$

where $\lambda_{\ell,j-1}$ is, for $\ell = 1, \ldots, g$ and $j \in \mathbb{N}$, the coefficient of $z^{j-1}$ in the Taylor series of $\omega_\ell/dz$ in $V$ (see, for instance, Farkas–Kra [3]), the equality 2.5.1 can be rewritten as:

$$\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_g \end{pmatrix} = -\begin{pmatrix} \frac{\lambda_{1,0}}{1} & \frac{\lambda_{1,1}}{2} & \cdots & \frac{\lambda_{1,g-1}}{g} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_{g,0}}{1} & \frac{\lambda_{g,1}}{2} & \cdots & \frac{\lambda_{g,g-1}}{g} \end{pmatrix}^{-1} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_g \end{pmatrix}. \tag{2.6.1}$$

Note now that, for $\ell = 1, \ldots, g$, the coefficients of the $\ell$-th row of the square matrix whose inverse appears in this equality are precisely the coefficients of $z, z^2, \ldots, z^g$ in the Taylor series of $\phi_\ell$ at $\infty$, whence we easily deduce that both members of 2.6.1 are of the form:

$$-\begin{pmatrix} z + z^{g+1}h_1 \\ \vdots \\ z^g + z^{g+1}h_g \end{pmatrix},$$

for some $h_1, \ldots, h_g \in O(V)$. ∎

LEMMA 2.7: *Let $\delta = \sum_{n=1}^{\infty} p_n$ be a positive infinite divisor in $\mathcal{V}'$ such that $\sum_{n=1}^{\infty} z(p_n)^{g+1}$ converges absolutely, and let $f_n$ be a $\Delta$-simple function in $G_\infty$ with $p_n$ as divisor for every $n \in \mathbb{N}$. If $D$ is a coordinate disk in $V$ and if $h$ is a holomorphic function in a neighbourhood of $\overline{D}$ having a zero at $\infty$, then the series $\sum_{n=1}^{\infty} \int_{\partial D} h d\log f_n$ converges absolutely too.*

*Proof:* By 2.5.2,

$$d\log f_n = \theta_{p_n\infty} + \sum_{j=1}^{g} \varphi_j(p_n)\theta_j,$$

and so, assuming if necessary that all the $p_n$ are in $D$,

$$\sum_{n=1}^{\infty} \int_{\partial D} h d\log f_n = 2\pi i \sum_{n=1}^{\infty} \left( h(p_n) + \sum_{j=1}^{g} \frac{1}{j!} \frac{d^j h}{dz^j}(\infty)\varphi_j(p_n) \right),$$

and by Remark 2.3 and Lemma 2.6, the hypothesis on $\delta$ implies that this last series converges absolutely if and only if the series

$$\sum_{n=1}^{\infty} \left( h(p_n) - \sum_{j=1}^{g} \frac{d^j h}{dz^j}(\infty) \frac{z(p_n)^j}{j!} \right)$$

does.

Observe now that

$$\sum_{j=1}^{g} \frac{d^j h}{dz^j}(\infty) \frac{z^j}{j!}$$

is precisely the initial polynomial of degree $g$ in the Taylor series of $h$ at $\infty$, from which, again by our assumption on the convergence of $\sum_{n=1}^{\infty} z(p_n)^{g+1}$ and by Remark 2.3, one obtains the desired conclusion. ∎

*Remark 2.8:* It is convenient to choose a point $p_0 \in \mathcal{V} - \overline{V}$, fixed from now on, in order to have the possibility of normalizing functions and differentials in $\mathcal{V}'$ by imposing on them, respectively, the conditions of taking the value 1 and having a zero at $p_0$. We shall also suppose, since it will be useful later, that $p_0$ is not a Weierstrass point.

LEMMA 2.9: *Given $f_1 \in O(W - \{\infty\})$, for some open neighbourhood $W$ of $\infty$, there exists $f \in O(\mathcal{V}' - \{p_0\})$, having a pole of order $\leq g$ at $p_0$ (or no pole), and such that $f - f_1$ has no singularity at $\infty$.*

*Proof:* Let $\mathcal{O}$ be the sheaf of holomorphic functions in $\mathcal{V}$, $\mathcal{O}'$ be the sheaf on $\mathcal{V}$ whose sections in every open subset $U$ of $\mathcal{V}$ are the holomorphic functions in $U - \{p_0, \infty\}$, and consider the exact sequence of sheaves:

$$0 \to \mathcal{O} \to \mathcal{O}' \to \mathcal{F} \to 0,$$

where the second arrow represents the homomorphism defined by restriction of functions, and $\mathcal{F}$ is the quotient sheaf $\mathcal{O}'/\mathcal{O}$. Then, since $H^1(\mathcal{V}, \mathcal{O}') = 0$, one obtains from the corresponding cohomology exact sequence that:

$$0 \to \mathbb{C} \to \Gamma(\mathcal{V}, \mathcal{O}') \to \Gamma(\mathcal{V}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{O}) \to 0,$$

where we have used the standard notations. Let now $u$ be a coordinate in some open neighbourhood of $p_0$ such that $u(p_0) = 0$, and note that the quotient classes in $\Gamma(\mathcal{V}, \mathcal{F})/\operatorname{Im}(\Gamma(\mathcal{V}, \mathcal{O}'))$ of the $g$ sections of $\mathcal{F}$ in $\mathcal{V}$ whose values at $p_0$ are defined by the germs of $1/u, 1/u^2, \ldots, 1/u^g$, and whose values at $\infty$ are all 0, are linearly independent over $\mathbb{C}$ because of the election of $p_0$ as a non-Weierstrass point. Hence, these classes span the said quotient space (take into account that the dimension over $\mathbb{C}$ of $H^1(\mathcal{V}, \mathcal{O})$ is $g$), from which results easily the conclusion of the statement. ∎

Let $U$ and $D$ be coordinate disks in $V$, with $\overline{U} \subset D$, and let $\mathcal{U}$ be the complementary in $\mathcal{V}$ of $\overline{U}$. Let $f$ be a holomorphic function in $\mathcal{U}$, and $\theta$ be a holomorphic differential in $D - \{\infty\}$. As usual, we shall represent with the notation $(f, \theta)$ the pairing of $f$ and $\theta$ given by $\int_{\partial D_1} f\theta$, where $D_1$ is another coordinate disk in $D$ such that $\overline{U} \subset D_1$ (and, of course, the integral is independent of the $D_1$ considered verifying these conditions). We shall also use the analogous notation $(\omega, h)$ in order to denote the similarly defined pairing of $\omega \in \Omega(\mathcal{U})$ (the space of holomorphic differentials in $\mathcal{U}$) and a holomorphic function $h$ in $D - \{\infty\}$.

Let $O_0(\mathcal{U})$ be the subspace of $O(\mathcal{U})$ formed by the functions with a zero at $p_0$, and $\Omega_0(\mathcal{U})$ be the subspace of $\Omega(\mathcal{U})$ defined by the differentials with a zero of order $\geq g$ at $p_0$. The following lemma related with these spaces is a consequence of corollary 2 of theorem 9 in Royden [7]. In its statement, and in the sequel when necessary, we consider $\Omega_0(\mathcal{U})$ (and also $O_0(\mathcal{U})$) as a topological vector space, with the topology of uniform convergence in compact subsets of $\mathcal{U}$. Since it is linearly and topologically isomorphic to $O(\mathcal{U})$, with its analogous natural topology, it is a nuclear locally convex space.

LEMMA 2.10: *If $U$ and $\mathcal{U}$ are as above, for every continuous linear functional $\psi$ on $\Omega_0(\mathcal{U})$, there exists a holomorphic function $h$ in some open neighbourhood of $\overline{U}$, vanishing at $\infty$, and such that $\psi(\omega) = (\omega, h)$ for every $\omega \in \Omega_0(\mathcal{U})$.*

*Proof:* Let $\omega_0$ be a holomorphic differential in $\mathcal{V}'$, without zeros in $\mathcal{V}' - \{p_0\}$, and with a zero of order $g-1$ at $p_0$, and consider the linear topological isomorphism $T$, from $O_0(\mathcal{U})$ onto $\Omega_0(\mathcal{U})$, given by the multiplication by $\omega_0$. By the mentioned theorem of Royden, there exists a coordinate disk $D$ in $V$, with $\overline{U} \subset D$, and a holomorphic differential $\theta$ in $D$ such that if $\omega \in \Omega_0(\mathcal{U})$, then

$$\psi(\omega) = (\psi \circ T)(\omega/\omega_0) = (\omega/\omega_0, \theta) = (\omega, \theta/\omega_0).$$

Apply now Lemma 2.9 to $f_1 = \theta/\omega_0$ in order to obtain $f \in O(\mathcal{V}' - \{p_0\})$, with possibly a pole of order $\leq g$ at $p_0$, and such that $h = f + \theta/\omega_0$ vanishes at $\infty$, and note that by the residue theorem it is also true that $\psi(\omega) = (\omega, h)$ for every $\omega \in \Omega_0(\mathcal{U})$. ∎

To obtain, from convergent series $\sum_{n=1}^{\infty} d \log f_n$, with $f_n \in M(\mathcal{V}')$, convergent products $\prod_{n=1}^{\infty} f_n$, that is, to choose the multiplicative constant corresponding to each $f_n$ in such a way that $\prod_{n=1}^{\infty} f_n$ converges, it is useful to introduce the following:

*Definition 2.11:* We shall say that a function $f \in M(\mathcal{V}')$, having no zero or pole at the point $p_0$, is normalized, if $f(p_0) = 1$.

If $f$ has a zero or a pole at $p_0$, we require no condition on $f$ to be normalized, i.e. every such $f$ is normalized.

Observe that if $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$, with $f_n \in M(\mathcal{V}')$ for every $n \in \mathbb{N}$, then the last possibility in this definition can only hold for a finite number of factors.

If $f$ is a normalized $\Delta$-simple function in $G_\infty$, which has all its zeros and all its poles in $\Delta - \{p_0\}$, then $f$ is uniquely determined by its divisor. Note also that if $\sum_{n=1}^{\infty} d \log f_n$ converges normally in $\mathcal{V}'$, with $f_n \in M(\mathcal{V}')$ normalized for every $n \in \mathbb{N}$, then $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$ too.

THEOREM 2.12: *Let $\delta = \sum_{n=1}^{\infty} p_n$ be a positive infinite divisor in $\mathcal{V}'$, and, for every $n \in \mathbb{N}$, let $f_n$ be a normalized $\Delta$-simple function in $G_\infty$ with $p_n$ as divisor. Then, the following conditions on $\delta$ are equivalent:*

(1) *$\sum_{n=1}^{\infty} z(p_n)^{g+1}$ is absolutely convergent.*

(2) *The product* $\prod_{n=1}^{\infty} f_n$ *converges normally in* $\mathcal{V}'$.

(3) *There exists a Weierstrass product associated with* $\delta$ *with factors in* $K_\infty$.

*Proof:* (1)⇒(2). It is sufficient to see that $\sum_{n=1}^{\infty} d\log f_n$ converges normally in every open subset $\mathcal{U}$ of $\mathcal{V}'$ defined as in Lemma 2.10. Let $\zeta_n$ be, for every $n \in \mathbb{N}$, a holomorphic differential in $\mathcal{V}$ such that $d\log f_n + \zeta_n$ has at $p_0$ a zero of order $\geq g$. Then, there exists $n_0 \in \mathbb{N}$ such that $d\log f_n + \zeta_n \in \Omega_0(\mathcal{U})$, for every $n \geq n_0$. By Lemmas 2.7 and 2.10, $\sum_{n=n_0}^{\infty} \psi(d\log f_n + \zeta_n)$ converges absolutely for every continuous linear functional $\psi$ in $\Omega_0(\mathcal{U})$, and so, since this locally convex space is nuclear, we deduce (see for instance proposition 4.2.2 in Pietsch [6]) that $\sum_{n=n_0}^{\infty} d\log f_n + \zeta_n$ converges normally in $\mathcal{U}$, from which it results, as is easily seen, that the same holds for $\sum_{n=1}^{\infty} d\log f_n$ (consider the vector in $\mathbb{C}^g$ whose components are the integrals of $d\log f_n + \zeta_n$ along $A_1, \ldots, A_g$, and take into account that $\zeta_n$ is a linear combination with coefficients in $\mathbb{C}$ of $\omega_1, \ldots, \omega_g$, for every $n \in \mathbb{N}$, where $\omega_1, \ldots, \omega_g$ are as explained just after Proposition 2.5).

(2)⇒(3). Evident.

(3)⇒(1). Suppose that $\prod_{n=1}^{\infty} F_n$ is a Weierstrass product associated with $\delta$ with factors in $K_\infty$. By Proposition 2.5, and getting rid if necessary of some of them, we can suppose that all the functions $F_n$ are $\Delta$-simple. Therefore, by 2.5.2, one has in $\Delta$:

$$d\log F_n = \theta_{p_n\infty} + \sum_{j=1}^{g} \varphi_j(p_n)\theta_j,$$

whence, by multiplying by $z^{g+1}$ and integrating along $\partial D$, where $D$ is a coordinate disk in $V$, we obtain that the series

$$\sum_{n=1}^{\infty} \int_{\partial D} z^{g+1} d\log F_n = 2\pi i \sum_{n=1}^{\infty} z(p_n)^{g+1}$$

is absolutely convergent. ∎

Let $(p_n)$ be as above. Then, since the convergence of $\sum_{n=1}^{\infty} |z(p_n)|^{g+1}$ is, as we have seen, an equivalent condition to the existence of a Weierstrass product with factors in $K_\infty$ associated with $(p_n)$, it seems natural to investigate the convergence of similar series with lower values of the exponent. Let $f_n$ be as in Theorem 2.12, and $P_n(1/z)$ be, for every $n \in \mathbb{N}$, the unique polynomial in $1/z$, without independent term, such that $f_n = \tilde{f}_n e^{P_n(1/z)}$, for some $\tilde{f}_n \in M(V)$ (we can call $P_n(1/z)$ **the singularity exponent polynomial** of $f_n$). Then,

$$d\log f_n = d\log \tilde{f}_n - \frac{1}{z^2} P_n'(1/z)dz,$$

and comparing with 2.5.2 (with $p = p_n$), we deduce that:

$$P_n(1/z) = -\left(\frac{\varphi_1(p_n)}{z} + \frac{\varphi_2(p_n)}{2z^2} + \cdots + \frac{\varphi_g(p_n)}{gz^g}\right).$$

Note as a consequence that, by Lemma 2.6, the series of the singularity exponent polynomials of the $f_n$ may not converge absolutely coefficientwise despite the normal convergence in $\mathcal{V}'$ of $\prod_{n=1}^{\infty} f_n$. In fact, this lemma shows that for each $j$ fixed in $\{1, \ldots, g\}$, the series of the coefficients of $1/z^j$ in these polynomials converges absolutely if and only if $\sum_{n=1}^{\infty} |z(p_n)|^j$ does. The following corollary presents several conditions equivalent to the convergence of this last series for the value 1 of $j$.

COROLLARY 2.13: *Let $\delta$ and $f_n$ be as in the theorem. Then, the following conditions on $\delta$ are equivalent:*

(1) *$\sum_{n=1}^{\infty} z(p_n)$ is absolutely convergent.*
(2) *$\sum_{n=1}^{\infty} \theta_{p_n\infty}$ converges normally in $\mathcal{V}'$.*
(3) *The series $\sum_{n=1}^{\infty} \int_{B_j} \theta_{p_n\infty}$ is absolutely convergent, for $j = 1, \ldots, g$.*
(4) *The series $\sum_{n=1}^{\infty} \varphi_j(p_n)$ is absolutely convergent, for $j = 1, \ldots, g$.*
(5) *Each series of coefficients of the singularity exponent polynomials of the $f_n$ is absolutely convergent.*

*Furthermore, any of these conditions implies that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$.*

*Proof:* (1)⇒(2). Theorem 2.12 implies that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$, and, by Lemma 2.6 and Remark 2.3, $\sum_{n=1}^{\infty} |\varphi_j(p_n)| < +\infty$, for $j = 1, \ldots, g$. Therefore, by 2.5.2, $\sum_{n=1}^{\infty} \theta_{p_n\infty}$ converges normally in $\mathcal{V}'$.

(2)⇒(3). Evident.

(3)⇒(4). By 2.5.1.

(4)⇒(5) and (5)⇒(1). See the observations previous to this corollary. ∎

Note that although we had made the assumption that $g > 0$, Theorem 2.12 is true for $g = 0$, and reduces in this case to the trivial equivalence, for a divergent sequence $(a_n)$ in $\mathbb{C}$, between the condition $\sum 1/|a_n| < +\infty$, with the sum extended to the nonzero terms of $(a_n)$, and the condition that the product $\prod_{n=1}^{\infty} \lambda_n(z - a_n)$ converges normally in $\mathbb{C}$, for some sequence $(\lambda_n)$ in $\mathbb{C}^*$. The following generalization of Theorem 2.12 deals with Weierstrass products having factors with singularity exponent polynomials of larger degree than $g$, and also generalizes a well known result in the case $g = 0$.

THEOREM 2.14: *Let $\delta$ be as in Theorem 2.12, and $\ell \in \mathbb{N}$ be $\geq g$. Then:*

1. *The series $\sum_{n=1}^{\infty} z(p_n)^{\ell+1}$ converges absolutely if and only if there exists a Weierstrass product $\prod_{n=1}^{\infty} f_n$ associated with $\delta$, with $f_n \in \overline{K}_{\infty}$ having singularity exponent polynomial of degree $\leq \ell$ for every $n \in \mathbb{N}$.*
2. *If any of the conditions in (1) holds, the singularity exponent polynomial of $f_n$ can be chosen to be of the form*

$$Q_n(1/z) + \frac{z(p_n)^{g+1}}{(g+1)z^{g+1}} + \cdots + \frac{z(p_n)^{\ell}}{\ell z^{\ell}},$$

*where $Q_n$ is a polynomial of degree $\leq g$, for every $n \in \mathbb{N}$.*

*Proof:* (1) If there is a Weierstrass product as in the statement, then by reasoning as in the proof of (3)⇒(1) in Theorem 2.12, one deduces that $\sum_{n=1}^{\infty} |z(p_n)|^{\ell+1} < +\infty$.

Conversely, assume the convergence of this series and observe that as the case, $\ell = g$, is part of Theorem 2.12, we can also suppose that $\ell \geq g+1$. Consider a $\Delta$-simple function $F_n \in G_{\infty}$ having $p_n$ as divisor, and let $H_n \in A(\mathcal{V}')$ (notation as in Theorem 1.1) be, for every $n \in \mathbb{N}$, such that $\prod_{n=1}^{\infty} F_n e^{H_n}$ converges normally in $\mathcal{V}'$ (which exists by Theorem 1.2). Use now, for instance, the Riemann–Roch theorem to obtain $h_n \in A(\mathcal{V}')$ such that the coefficients of $1/z^{g+1}, 1/z^{g+2}, \ldots, 1/z^{\ell}$ in its Laurent series in $V'$ vanish, and such that $\mathrm{ord}_{\infty}(H_n - h_n) \geq -\ell$, and note that the proof will be finished if we see that $\sum_{n=1}^{\infty} h_n$ converges normally in $\mathcal{V}'$. By the theorem of Royden used in the proof of Lemma 2.10, it will suffice to demonstrate that if $\alpha$ is any holomorphic function in some coordinate disk $W$ in $V$, with a zero of order $\geq g$ at $\infty$, and if $D$ is a coordinate disk in $W$, then $\sum_{n=1}^{\infty} \int_{\partial D} h_n \alpha dz$ converges absolutely. This is clear if $\alpha$ is of the particular form $z^j$, with $g \leq j \leq \ell - 1$, because of the assumption on the coefficient of $1/z^{j+1}$ in the Laurent series of $h_n$ (which coincides save for the factor $\frac{1}{2\pi i}$ with $\int_{\partial D} h_n z^j dz$), and so we can reason assuming, further, that $\alpha$ has a zero of order $\geq \ell$ at $\infty$. Let $W$ and $D$ be as above for the considered $\alpha$, and let $\beta \in O(W)$, with a zero of order $\geq \ell + 1$ at $\infty$, be such that $d\beta/dz = \alpha$. Then,

$$\int_{\partial D} h_n \alpha dz = -\int_{\partial D} \beta dh_n = 2\pi i \beta(p_n) - \int_{\partial D} \beta(dH_n + d\log F_n),$$

where the first equality is an integration by parts and the other is a consequence of the coincidence of the coefficients of $dz/z^j$ in the Laurent series in $V'$ of $dh_n$ and of

$d\log F_n + dH_n$, for every $j > \ell+1$. Finally, take into account that the convergence of $\prod_{n=1}^{\infty} F_n e^{H_n}$ implies the absolute convergence of $\sum_{n=1}^{\infty}\int_{\partial D}\beta(dH_n + d\log F_n)$, and that $\sum_{n=1}^{\infty}|\beta(p_n)|$ converges by Remark 2.3, in order to obtain the desired conclusion.

*Proof of (2):* From $\sum_{n=1}^{\infty}|z(p_n)|^{\ell+1} < +\infty$, one deduces, as in the usual proof of the classical Weierstrass factorization theorem, that if

$$\overline{f}_n = \left(1 - \frac{z(p_n)}{z}\right) e^{\frac{z(p_n)}{z} + \frac{z(p_n)^2}{2z^2} + \cdots + \frac{z(p_n)^\ell}{\ell z^\ell}},$$

then $\prod_{n=1}^{\infty}\overline{f}_n$ converges normally in $V'$. Hence, $d\log\overline{f}_n - d\log f_n$ being holomorphic in $V'$, it follows that if $D$ is a coordinate disk in $V$, then $\sum_{n=1}^{\infty}\int_{\partial D} z^j(d\log\overline{f}_n - d\log f_n)$ converges absolutely for every $j \in \mathbb{N}$, whence we obtain that the series of the coefficients of $z^j$ in the singularity exponent polynomials of the $\overline{f}_n$, and the analogous series for the $f_n$, differ in an absolutely convergent one, say $\sum_{n=1}^{\infty}\mu_{jn}$, for every $j \in \mathbb{N}$. Finally, to finish the proof, note that if $h_j \in A(\mathcal{V}')$ has singular part equal to $1/z^j$ at $\infty$, for $g+1 \le j \le \ell$, then $\sum_{n=1}^{\infty}\mu_{jn}h_j$ converges normally in $\mathcal{V}'$. ∎

Besides the generalization provided by Theorem 2.14, we want also to generalize partially Theorem 2.12 and Corollary 2.13, by considering more than one sequence, of the type of the $(p_n)$ of above, in order to obtain, as an application, a result characterizing the infinite products, of functions in $M(\mathcal{V})$ with bounded degree, which converge normally in $\mathcal{V}'$. First note that, up to now, we have always considered a sequence of points in $\mathcal{V}'$; that is, we have supposed that each one of these points is different from $\infty$. Of course, the differential $\theta_{p_n\infty}$ and the normalized function $f_{p_n}(= f_n)$ appearing in the statements of Theorem 2.12 and Corollary 2.13 seem to have no meaning when $p_n = \infty$, but we can adopt the arrangement that $\theta_{\infty\infty} = 0$ and that $f_\infty = 1$ (justified by the fact that for every sequence $(a_n)$ in $\mathcal{V}'$ converging to $\infty$, the corresponding sequences $(\theta_{a_n\infty})$ and $(f_{a_n})$ converge uniformly in compact subsets of $\mathcal{V}'$, respectively to 0 and 1, as is not very hard to check). With this arrangement, if we extend in the obvious way the definition of Weierstrass products, the correctness of the following becomes evident:

*Remark 2.15:* For a sequence $(p_n)$ of points in $\mathcal{V}$ converging to $\infty$, the analogous conclusions to those of Theorem 2.12 and Corollary 2.13 also hold.

The above-referred useful generalization for proving a result about infinite products of functions in $M(\mathcal{V})$, consists in replacing the sequence of points $(p_n)$ by a sequence of finite divisors. Given $k \in \mathbb{N}$, let $D_k$ be the set of divisors in $\mathcal{V}$ of the form $a_1 + \cdots + a_r$, with $a_1, \ldots, a_r \in V$ and $r \leq k$, and let $\sigma_1, \ldots, \sigma_k$ be the functions defined, for every $d = a_1 + \cdots + a_r \in D_k$ and for $\ell = 1, \ldots, k$, by $\sigma_\ell(d) = z(a_1)^\ell + \cdots + z(a_r)^\ell$. Then, the subset of $D_k$ formed by the divisors of degree $k$ can be considered as the $k$-th symmetric product $V^{(k)}$, and can be naturally endowed with a $k$-dimensional complex manifold structure, being well known that in this manifold the functions $\sigma_1, \ldots, \sigma_k$ are coordinates, i.e. the restriction to $V^{(k)}$ of the mapping $\Gamma_k\colon D_k \to \mathbb{C}^k$, with components $\sigma_1, \ldots, \sigma_k$, is a holomorphic isomorphism of $V^{(k)}$ with an open subset of $\mathbb{C}^k$ (see, for instance, Gunning [4]).

If we denote by $\mathcal{D}_k$ the set of divisors of the type $\delta = d - d'$, with $d, d' \in D_k$, we shall also set $\sigma_\ell(\delta) = \sigma_\ell(d) - \sigma_\ell(d')$, for $\ell = 1, \ldots, k$, and $\Gamma_k(\delta) = (\sigma_1(\delta), \ldots, \sigma_k(\delta)) \in \mathbb{C}^k$. The following notation will be used too: for every such $\delta = d - d'$, if $d = a_1 + \cdots + a_r$ and $d' = b_1 + \cdots + b_s$, then $\theta_{\delta\infty}$ will denote briefly $\theta_{a_1\infty} + \cdots + \theta_{a_r\infty} - (\theta_{b_1\infty} + \cdots + \theta_{b_s\infty})$, and for every $\phi \in O(V^{(k)})$ and $\varphi \in O(V)$, we shall also represent by $\phi$ and $\varphi$ their naturally defined extensions to $\mathcal{D}_k$, i.e. $\phi(\delta) = \phi\left(d + (k-r)\infty\right) - \phi\left(d' + (k-s)\infty\right)$ and $\varphi(\delta) = \varphi(a_1) + \cdots + \varphi(a_r) - (\varphi(b_1) + \cdots + \varphi(b_s))$, with $\delta$ as above.

Let $(\delta_n)$ be a sequence in $\mathcal{D}_k$ tending to $\infty$ (in the sense of (2) in Warning 2.2), and note that since the convergence of $\sum_{n=1}^{\infty} \|\Gamma_k(\delta_n)\|$ (where $\|\ \|$ denotes, for instance, the usual norm in $\mathbb{C}^k$) implies the convergence of $\sum_{n=1}^{\infty} |\phi(\delta_n)|$ for every holomorphic function $\phi$ in $V^{(k)}$ having a zero at $k\infty$ (and hence the convergence of $\sum_{n=1}^{\infty} |\varphi(\delta_n)|$ for every $\varphi \in O(V)$ vanishing at $\infty$), then by applying the same device as in the proofs of Theorem 2.12 and Corollary 2.13, one can demonstrate without difficulty the following:

THEOREM 2.16: *Let $(\delta_n)$ be a sequence in $\mathcal{D}_k$ tending to $\infty$. If $f_{\delta_n}$ is the normalized $\Delta$-simple function in $G_\infty$ having as divisor the restriction of $\delta_n$ to $\mathcal{V}'$, then the following conditions are equivalent:*

(1) *$\sum_{n=1}^{\infty} \|\Gamma_k(\delta_n)\|$ is convergent.*
(2) *$\sum_{n=1}^{\infty} \theta_{\delta_n\infty}$ converges normally in $\mathcal{V}'$.*
(3) *The series $\sum_{n=1}^{\infty} \int_{B_j} \theta_{\delta_n\infty}$ is absolutely convergent, for $j = 1, \ldots, g$.*
(4) *The series $\sum_{n=1}^{\infty} \varphi_j(\delta_n)$ is absolutely convergent, for $j = 1, \ldots, g$.*
(5) *Each series of coefficients of the singularity exponent polynomials (in a*

*sense similar to that of Corollary 2.11) of the $f_{\delta_n}$ is absolutely convergent.*

*Furthermore, any of these conditions implies that $\prod_{n=1}^{\infty} f_{\delta_n}$ converges normally in $\mathcal{V}'$.*

In the particular case in which the $\delta_n$ are principal divisors in $\mathcal{V}$, one can study, with the help of Theorem 2.16, the convergence of certain products of functions in $M(\mathcal{V})$. Note, for later use, that if $\delta$ is the divisor of a function $\alpha \in M(\mathcal{V})$, and is supported in $\Delta$, then

$$d\log\alpha = \theta_{\delta\infty} + \sum_{j=1}^{g}\left(\int_{A_j} d\log\alpha\right)\omega_j. \tag{2.16.1}$$

In particular, if we assume further that $\alpha$ is $\Delta$-simple, then

$$d\log\alpha = \theta_{\delta\infty}. \tag{2.16.2}$$

THEOREM 2.17: *Let $(\alpha_n)$ be a sequence of normalized functions in $M(\mathcal{V})$, with degrees bounded by $k \in \mathbb{N}$. Then, $\prod_{n=1}^{\infty} \alpha_n$ converges normally in $\mathcal{V}'$ if and only if the sequence $(\delta_n)$ of the divisors of the $\alpha_n$ tends to $\infty$ and $\sum_{n=1}^{\infty} \|\Gamma_k(\delta_n)\|$ converges.*

*Proof:* Assume that the product of the $\alpha_n$ converges normally in $\mathcal{V}'$. Then, the sequence of the $\delta_n$ tends to $\infty$ and so, by getting rid if necessary of a finite number of its terms, we can consider a coordinate disk $D$ in $V$ containing the zeros and poles of all the $\alpha_n$. We can also suppose, by Proposition 2.5, that $\alpha_n$ is $\Delta$-simple for every $n \in \mathbb{N}$. Hence, multiplying by $z^\ell$ the series $\sum_{n=1}^{\infty} d\log\alpha_n$, and then integrating along the boundary of $D$, one arrives easily, using 2.16.2, at the conclusion that $\sum_{n=1}^{\infty} |\sigma_\ell(\delta_n)|$ converges for $\ell = 1, \ldots, k$.

Conversely, suppose that these series converge and that $(\delta_n)$ tends to $\infty$. The idea of what follows is to see that these hypotheses imply also that there exists $n_0 \in \mathbb{N}$ such that $\alpha_n$ is $\Delta$-simple for every $n \geq n_0$, from which by 2.16.2 and by Theorem 2.16 one deduces easily the normal convergence in $\mathcal{V}'$ of $\prod_{n=1}^{\infty} \alpha_n$

By integrating both members of 2.16.1 (with $\alpha = \alpha_n$) along $B_l$, for $l = 1, \ldots, g$, and expressing the resulting equality by means of matrices, one has:

$$\begin{pmatrix} \int_{B_1} \theta_{\delta_n\infty} \\ \vdots \\ \int_{B_g} \theta_{\delta_n\infty} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \int_{B_1}\omega_1 & \cdots & \int_{B_1}\omega_g \\ 0 & 1 & \cdots & 0 & \int_{B_2}\omega_1 & \cdots & \int_{B_2}\omega_g \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \int_{B_g}\omega_1 & \cdots & \int_{B_g}\omega_g \end{pmatrix} \begin{pmatrix} \int_{B_1} d\log\alpha_n \\ \vdots \\ \int_{B_g} d\log\alpha_n \\ -\int_{A_1} d\log\alpha_n \\ \vdots \\ -\int_{A_g} d\log\alpha_n \end{pmatrix}.$$

Recall now that, as is well known, the columns of the $g \times 2g$ matrix appearing in this equality are linearly independent over $\mathbb{R}$, and so generate a lattice $L$ in $\mathbb{C}^g$ (such that $\mathbb{C}^g/L$ is the jacobian variety of $\mathcal{V}$), and note on the other hand that the point of $\mathbb{C}^g$ defined by the first member comes close to 0 for large values of $n$. Since this can only happen if for all but possibly a finite number of values of $n$ this point is 0, we obtain easily what we wanted to prove. ∎

*Remark 2.18:* Up to now we have always considered a point $\infty \in \mathcal{V}$ which is not of Weierstrass, but we want also to explain briefly how the preceding theory can be transcribed to the general case in which $\infty$ may be a Weierstrass point.

Let $r_1 < \cdots < r_g$ be the Weierstrass gaps at $\infty$, $\rho = \rho_1 < \cdots < \rho_g$ be the nongaps belonging to $\{1, \ldots, 2g\}$, and $\theta_j$ be the unique holomorphic differential in $\mathcal{V}'$ with null integrals over $A_1, \ldots, A_g$ and with singular part at $\infty$ equal to $dz/z^{r_j+1}$, for $j \in \{1, \ldots, g\}$. For every $p \in \Delta' - p_0$, let $f_p$ be the normalized function defined by

$$(2.18.1) \qquad d\log f_p = \theta_{p\infty} + \sum_{j=1}^{g} \varphi_j(p)\theta_j,$$

where $(\varphi_1(p), \ldots, \varphi_g(p)) \in \mathbb{C}^g$ is such that the second member of 2.18.1 has null integrals along $B_1, \ldots, B_g$ as well. Note the analogy with 2.5.2, and consider the subgroup $G_\infty$, of $G(\mathcal{V}')$, generated over $M^*(\mathcal{V})$ by all these functions $f_p$, and the field $K_\infty$ formed by the quotients of finite sums of functions in $G_\infty$ with non-identically-zero denominator. As it is not difficult to see, these definitions coincide with the previous ones if $\infty$ is not a Weierstrass point (see Cutillas [2]).

Let $\delta = \sum_{n=1}^{\infty} p_n$ be a positive infinite divisor in $\mathcal{V}'$ and, for every $n \in \mathbb{N}$, let $f_n$ be as the $f_p$ of above with $p$ replaced by $p_n$. Consider the two conditions on $\delta$:

(1) $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$.

(2) $\sum_{n=1}^{\infty} z(p_n)^{\rho}$ is absolutely convergent.

Then, reasoning as in (3)⇒(1) of the proof of Theorem 2.12, one deduces easily that (1)⇒(2). Conversely, observe that similar arguments to those in the proof of Lemma 2.6, lead to the conclusion that $\varphi_j + z^{r_j}$ plus a linear combination, with coefficients in $\mathbb{C}$, of $z^{\rho_1}, \ldots, z^{\rho_g}$, has at $\infty$ a zero of order $\geq 2g$ for every $j \in \{1, \ldots, g\}$, and note also that this implies the analogue of Lemma 2.7 with $\rho$ instead of $g+1$, whereupon we can obtain as in the proof of the said theorem that (2)⇒(1) too. Theorem 2.12 is thus generalized, and from this generalization one can deduce that Corollary 2.13, (an analogue of) Theorem 2.14, Theorem 2.16 and Theorem 2.17 are also valid without necessarily supposing that $\infty$ is not a Weierstrass point.

## 3. WF-fields

It was already explained in the Introduction what a WF-field is, but we want now to state formally and precisely the meaning of this concept.

*Definition 3.1:* A subfield $K$ of $M(\mathcal{V}')$ will be called a WF-field (in $\mathcal{V}'$), if it contains $M(\mathcal{V})$, and if for every positive infinite divisor $\delta$ in $\mathcal{V}'$ there exists a Weierstrass product with factors in $K$ associated with $\delta$.

It is a clear consequence of this definition, and of the generalization for compact Riemann surfaces of the Weierstrass factorization theorem (mentioned in the Introduction), that $M(\mathcal{V}')$ is a WF-field in $\mathcal{V}'$, and we also know (Theorem 1.2) that $\overline{K}_{\infty}$ is another example of a WF-field in $\mathcal{V}'$.

An elementary but useful observation about this type of field is the following:

PROPOSITION 3.2: *A subfield $K$ of $M(\mathcal{V}')$ is a WF-field if and only if for every sequence $(\delta_n)$ of finite divisors in $\mathcal{V}'$ which tends to $\infty$, there exists a sequence of functions $(f_n)$ in $K$, such that for every $n \in \mathbb{N}$ the divisor of $f_n$ is $\delta_n$, and such that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$.*

From now on, for every subfield $K$ of $M(\mathcal{V}')$, $H(K)$ will be the subgroup of $O(\mathcal{V}')$ formed by the functions $\tau$ such that $e^{\tau} \in K$. We shall soon see that these associated groups are fundamental in the theory of WF-fields.

LEMMA 3.3: *Let $K$ be a WF-field, and $(\alpha_n)$ be a sequence of nonzero functions of $A(\mathcal{V}')$, which converges uniformly in every compact subset of $\mathcal{V}'$ to $e^{\tau}$, with*

*$\tau \in O(\mathcal{V}')$. Then there is a sequence $(h_n)$ in $H(K)$ such that $\prod_{n=1}^{\infty} \alpha_n e^{h_n}$ converges normally in $\mathcal{V}'$.*

*Proof:* Since $(\alpha_n)$ converges uniformly in compact subsets of $\mathcal{V}'$ to an exponential function, its sequence of divisors $(\delta_n)$ tends to $\infty$ and we can assume, by getting rid if necessary of a finite number of them, that all the $\alpha_n$ are $\Delta$-simple. By Proposition 3.2, there exists a sequence of functions $f_n$, in $K$, such that the divisor of $f_n$ is the restriction of $\delta_n$ to $\mathcal{V}'$ for every $n \in \mathbb{N}$, and such that $\prod_{n=1}^{\infty} f_n$ converges normally in $\mathcal{V}'$, while by Proposition 2.5 it can be also supposed, without loss of generality, that $f_n$ is $\Delta$-simple, for every $n \in \mathbb{N}$. Therefore, being $\alpha_n$ and $f_n$ $\Delta$-simple functions having the same divisor (in $\mathcal{V}'$) for every $n \in \mathbb{N}$, their quotient must be an exponential belonging to $K$. ∎

In the following two definitions we explain some more useful terminology. The concept of W-field appearing in the second was already mentioned in the Introduction, and was introduced and investigated in Cutillas [2].

*Definition 3.4:* A generalized Weierstrass product in $\mathcal{V}'$ will be an infinite product, normally convergent in $\mathcal{V}'$, of the type $\prod_{n=1}^{\infty} f_n$, where $f_n \in O(\mathcal{V}')$ has either no zero in $\mathcal{V}'$ or a simple zero at a unique point of $\mathcal{V}'$, for every $n \in \mathbb{N}$.

A generalized Weierstrass product with factors in a subfield $K$ of $M(\mathcal{V}')$ will be a product as above, with all the $f_n$ belonging to $K$.

*Definition 3.5:* Let $K$ be a subfield of $M(\mathcal{V}')$. We shall say that it verifies the Weierstrass property (in $\mathcal{V}'$) or, in short, that it is a W-field, if it contains $M(\mathcal{V})$ and if for every finite divisor $\delta$ in $\mathcal{V}'$ there is a function in $K$ whose divisor is $\delta$.

Note that the class of W-fields contains that of WF-fields. Since all W-fields generated by functions with finite divisor were determined in Cutillas [2], we are interested in founding possible additional properties verified only by WF-fields: for instance, those appearing in the following:

THEOREM 3.6: *For a subfield $K$ of $M(\mathcal{V}')$, containing $M(\mathcal{V})$, the following conditions are equivalent:*

(1) *$K$ is a WF-field.*
(2) *$K$ is a W-field and $H(K)$ is a dense subgroup of $O(\mathcal{V}')$.*
(3) *Every function with infinite divisor in $O(\mathcal{V}')$ is a Weierstrass product with factors in $K$.*

(4) *Every function in* $O(\mathcal{V}')$ *is a generalized Weierstrass product with factors in* $K$.

*Proof:* (1)⇒(2). Let $\tau$ be any function in $O(\mathcal{V}')$, and let $(\alpha_n)$ be a sequence of nonzero functions in $A(\mathcal{V}')$ converging uniformly in every compact subset of $\mathcal{V}'$ to $e^\tau$ (Theorem 1.1). By Lemma 3.3, there exists a sequence $(h_n)$ in $H(K)$ such that $\prod_{n=1}^\infty \alpha_n e^{h_n}$ converges normally in $\mathcal{V}'$, whence we deduce that $(\alpha_n e^{h_n})$ converges uniformly to 1 in every compact subset of $\mathcal{V}'$, from which it results easily that $\tau$ is the uniform limit in compact subsets of $\mathcal{V}'$ of $(-h_n + 2\pi i k_n)$ for some sequence of integers $(k_n)$.

(2)⇒(1). Let $\delta = \sum_{n=1}^\infty p_n$ be a positive infinite divisor in $\mathcal{V}'$, and $(V_k)$ be a sequence of coordinate disks in $V$ with the radii of the $z(V_k)$ tending to 0, and such that all the $p_n$ are in $V_1$ and none of them is in $\bigcup_{k=1}^\infty \partial V_k$. By reordering the sequence $(p_n)$ and replacing, if necessary, $(V_k)$ by some subsequence, we can suppose that $p_1, \ldots, p_{n_1} \in V_1 - \overline{V}_2$, and that for $k \in \mathbb{N}$, $p_{n_k+1}, \ldots, p_{n_{k+1}} \in V_{k+1} - \overline{V}_{k+2}$, for some increasing sequence $(n_k)$ in $\mathbb{N}$. Given any $n \in \mathbb{N}$, let $k \in \mathbb{Z}^+$ be such that $n_k + 1 \leq n \leq n_{k+1}$, where we are taking $n_k = 0$ if $k = 0$ and $n_k$ as above if $k \in \mathbb{N}$, and let $f_n \in K$ be a $\Delta$-simple function having $p_n$ as divisor in $\mathcal{V}'$. Then, in some neighbourhood of $\mathcal{V} - V_{k+1}$, there is a uniform branch $\log f_n$ of the logarithm of $f_n$ and so, by Theorem 1.1, there exists $h_n \in H(K)$ such that

$$\|\log f_n + h_n\|_{\mathcal{V}-D_{k+1}} < \frac{2^{-k}}{n_{k+1} - n_k},$$

whence one easily obtains that $\prod_{n=1}^\infty f_n e^{h_n}$ converges normally in $\mathcal{V}'$.

(1)⇒(3). Let $f$ be a holomorphic function in $\mathcal{V}'$ with infinite divisor $\delta = \sum_{n=1}^\infty p_n$, and let $\prod_{n=1}^\infty f_n$ be a Weierstrass product associated with $\delta$ with factors in $K$. Then by multiplying, if necessary, $f_1$ by a suitable holomorphic function of $K$ without zeros in $\mathcal{V}'$, we can suppose further that $f \cdot (\prod_{n=1}^\infty f_n)^{-1} = e^\tau$, for some $\tau \in O(\mathcal{V}')$. Now, take into account that $H(K)$ is dense in $O(\mathcal{V}')$ in order to choose a sequence $(\tau_n)$, in $H(K)$, such that $\sum_{n=1}^\infty \tau_n$ converges normally in $\mathcal{V}'$ to $\tau$, and note that this implies that $\prod_{n=1}^\infty f_n e^{\tau_n}$ is also a Weierstrass product with factors in $K$ associated with $\delta$, and that one has $f = \prod_{n=1}^\infty f_n e^{\tau_n}$.

(3)⇒(1) Easy.

(1)⇒(4). Use that $H(K)$ is dense in $O(\mathcal{V}')$ as in the proof of (1)⇒(3).

(4)⇒(1). Given a divisor $\delta$ in $\mathcal{V}'$, consider a function $f \in O(\mathcal{V}')$ with divisor $\delta$, and a generalized Weierstrass product $\prod_{n=1}^\infty f_n$, with factors in $K$, convergent to $f$. Then, since $\prod_{n=1}^{\infty} f_n$ is normally convergent in $\mathcal{V}'$, the same must be true for $\prod_{\ell=1}^{\infty} f_{n_\ell}$, for every subsequence $(f_{n_\ell})$ of $(f_n)$, from which results the desired conclusion. ∎

A natural question about WF-fields which one may pose is: does there exist any minimal WF-field in $\mathcal{V}'$? Condition (2) of Theorem 3.6 suggests that the answer is negative, and the following corollary shows that in fact it is so.

COROLLARY 3.7: *There is no minimal WF-field in $\mathcal{V}'$.*

*Proof:* By Theorem 3.6, it suffices to see that there is no minimal dense subgroup of $O(\mathcal{V}')$. Note, first, that if $G$ is a dense subgroup of $O(\mathcal{V}')$ then, for every $p \in \mathbb{N}$, $pG$ is also a dense subgroup of $O(\mathcal{V}')$, and therefore, a minimal dense subgroup of $O(\mathcal{V}')$ would be a minimal dense $\mathbb{Q}$-vector subspace of $O(\mathcal{V}')$ too. Taking this into account, we can apply a standard argument which we include below for the sake of completeness. If there would exist such a $\mathbb{Q}$-vector subspace $H$ of $O(\mathcal{V}')$, we could consider a sequence $(h_n)$ of functions in $H$, linearly independent over $\mathbb{Q}$ and such that $\|h_n\|_{Q_n} \leq 1$ for every $n \in \mathbb{N}$, where $(Q_n)$ is an exhaustive sequence of compacts in $\mathcal{V}'$ (in the standard sense). Being $H$ minimal, all its $\mathbb{Q}$-hyperplanes would be closed and so (see, for instance, Schaefer [8]), all $\mathbb{Q}$-linear forms in $H$ would be continuous. In particular, a $\mathbb{Q}$-linear form $\phi$ in $H$ such that $\phi(h_n) = n$ for every $n \in \mathbb{N}$ would be continuous and so, for some compact subset $Q$ of $\mathcal{V}'$ and some constant $C > 0$, we would have $n \leq C\|h_n\|_Q$ for every $n \in \mathbb{N}$, which is impossible since $Q$ is contained in all but a finite number of the $Q_n$. ∎

Theorem 3.6, together with a result in Cutillas [2], permits us to determine all WF-fields in $\mathcal{V}'$ generated by functions with finite divisor, by means of certain group homomorphisms. That paper presented a natural way of associating every W-field $K$ in $\mathcal{V}'$ with a group homomorphism $\psi_K\colon \mathbb{C}^g \to O(\mathcal{V}')/H(K)$, which we recall briefly now, in two steps:

(1) Let $J(\Delta)$ be the quotient group of the group of finite divisors with degree 0 supported in $\Delta$, by the subgroup of divisors of $\Delta$-simple meromorphic functions in $\mathcal{V}$. Then, the mapping from $J(\Delta)$ into $\mathbb{C}^g$ such that the image of the quotient class, in $J(\Delta)$, of the divisor $\sum_{i=1}^{r} n_i(a_i - \infty)$, with $a_i \in \Delta'$ for $i = 1, \ldots, r$, is:

$$\left(\int_{B_1} \sum_{i=1}^{r} n_i \theta_{a_i\infty}, \ldots, \int_{B_g} \sum_{i=1}^{r} n_i \theta_{a_i\infty}\right),$$

is well defined and is in fact a group isomorphism from $J(\Delta)$ onto $\mathbb{C}^g$.

(2) For every $a \in \Delta'$ there exists a $\Delta$-simple $f_a \in K$, with divisor $a$ in $\mathcal{V}'$. Let $\lambda_1, \ldots, \lambda_g \in \mathbb{C}$ be such that $d \log f_a = \theta_{a\infty} + \sum_{j=1}^{g} \lambda_j \theta_j + dh_a$, where $h_a \in O(\mathcal{V}')$ and the notation is as in Section 2. Then, the correspondence $a - \infty \to \overline{h}_a$, where $\overline{h}_a$ is the quotient class of $h_a$ in $O(\mathcal{V}')/H(K)$, can be extended to an homomorphism, from the group of finite divisors with degree 0 supported in $\Delta$ onto $O(\mathcal{V}')/H(K)$, which is zero on the subgroup of divisors of $\Delta$-simple meromorphic functions in $\mathcal{V}$, and so defines an homomorphism $\psi_K$ from $J(\Delta)$ into $O(\mathcal{V}')/H(K)$, which by (1) can be thought of as defined on $\mathbb{C}^g$.

The above said result in Cutillas [2] is that the map $K \to \psi_K$ is a bijection $\Psi$ of the set of all W-fields in $\mathcal{V}'$ generated by functions with finite divisor, with the set of all homomorphisms $\psi$ from $\mathbb{C}^g$ into any possible quotient group $O(\mathcal{V}')/H$ of $O(\mathcal{V}')$ by a subgroup $H$ containing $\mathbb{C}$ (this subgroup coinciding with $H(K)$ if $\psi = \psi_K$). Note finally that every W-field in $\mathcal{V}'$ contains the W-field generated by its functions with finite divisor, and that the analogous assertion is true for WF-fields.

COROLLARY 3.8: *The restriction of $\Psi$ to the set of WF-fields in $\mathcal{V}'$ generated by functions with finite divisor is a bijection with the set of all homomorphisms $\psi$ from $\mathbb{C}^g$ into any possible quotient group $O(\mathcal{V}')/H$ of $O(\mathcal{V}')$ by a dense subgroup $H$ containing $\mathbb{C}$ (this subgroup coinciding with $H(K)$, if $\psi = \Psi(K)$ for some WF-field $K$ in $\mathcal{V}'$).*

*Remark 3.9:* Throughout this section, $\mathcal{V}'$ has been the complementary in $\mathcal{V}$ of a unique point $\infty$. If, instead of this, $\mathcal{V}'$ is of the form $\mathcal{V} - S$, for some nonvoid finite subset $S$ of $\mathcal{V}$, we can define the concept of WF-field in $\mathcal{V}'$ in a similar way. $M(\mathcal{V}')$ and the subfield, analogous to $\overline{K}_\infty$, generated by the functions with polynomic exponential singularities at the points of $S$, are likewise examples of WF-fields in $\mathcal{V}'$. The concepts of Weierstrass product and generalized Weierstrass product can also be easily generalized to this case and, by using the same type of arguments, analogues of Theorem 3.6 and Corollaries 3.7 and 3.8 can be proved.

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## References

[1] H. F. Baker, *Abel's Theorem and the Allied Theory Including the Theory of Theta Functions*, Cambridge University Press, 1897.

[2] P. Cutillas Ripoll, *Construction of certain function fields associated with a compact Riemann surface*, American Journal of Mathematics **106** (1984), 1423–1450.

[3] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer, Berlin, 1980.

[4] R. C. Gunning, *Lectures on Riemann surfaces. Jacobi varieties*, Princeton Mathematical Notes, 1972.

[5] P. Günther, *Über die eindeutigen funktionen von zwei durch eine algebraische gleichung verbundenen veränderlichen*, Crelle **109** (1892), 199–212.

[6] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer, Berlin, 1972.

[7] H. L. Royden, *Function theory on compact Riemann surfaces*, Journal d'Analyse Mathématique **18** (1967), 275–327.

[8] H. H. Schaefer, *Topological Vector Spaces*, Springer, Berlin, 1971.